Student No:	

Mathematics Extension 1

Year 12, Assessment task 3, Term 2 2023

General Instruction:

- Reading time – 10 minutes
- Working time – 120 minutes
- Write using black/blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks:

Section I – 10 marks (pages 1–3)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 4–7)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Class Teacher (please tick one name)

- ◯ Mr Berry
- ◯ Ms Fu
- ◯ Mr Ireland
- ◯ Ms Lee
- ◯ Mr Lin
- ◯ Mr Umakanthan
- ◯ Dr Vranešević

Question No.	1-10	11	12	13	14	Total	Total %
Marks	/10	/15	/15	/15	/15	/70	

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Which of the following is the correct expression for $\displaystyle\int \frac{dx}{\sqrt{36-x^2}}$?

   A) $\arccos\dfrac{x}{6}+c$;
   B) $\arccos 6x+c$;
   C) $\arcsin\dfrac{x}{6}+c$;
   D) $\arcsin 6x+c$.

2. Find the fifth term in the expansion of $\left(3x^2-\dfrac{1}{2x}\right)^8$.

   A) $\dbinom{8}{4}\left(\dfrac{3}{2}\right)^4 x^4$;
   B) $\dbinom{4}{8}\left(\dfrac{2}{3}\right)^4 x^{-4}$;
   C) $\dbinom{8}{4}\left(\dfrac{3}{2}\right)^4 x^{-4}$;
   D) $\dbinom{4}{8}\left(\dfrac{2}{3}\right)^4 x^4$.

3. A projectile has an initial velocity vector $\vec{v}=\begin{pmatrix}2\sqrt{3}\\2\end{pmatrix}$. Which of the following is the correct statement of its initial speed and angle of projection from the horizontal?

   A) $4\,\mathrm{m/s}$ at $30°$;
   B) $\sqrt{10}\,\mathrm{m/s}$ at $30°$;
   C) $4\,\mathrm{m/s}$ at $60°$;
   D) $\sqrt{10}\,\mathrm{m/s}$ at $60°$.

4. If $f(x)=e^{x+2}$ what is the inverse function $f^{-1}(x)$?

   A) $f^{-1}(x)=e^{y-2}$;
   B) $f^{-1}(x)=e^{y+2}$;
   C) $f^{-1}(x)=\log_e x-2$;
   D) $f^{-1}(x)=\log_e x+2$.

5. The radius of a sphere is increasing at the rate of 5 centimetres per minute. What is the rate of increase of the volume of the sphere, in cubic centimetres per minute, when the radius is 6 centimetres?

   A) 144π;

   B) 288π;

   C) 600π;

   D) 720π.

6. If $x+a$ is a factor of $4x^3+13x^2+ax$, then the value of a is:

   A) 3;

   B) 1;

   C) -3;

   D) -4.

7. If $\vec{a}=-2\hat{i}-\hat{j}$ and $\vec{b}=m\hat{i}-\hat{j}$, where m is a real constant, then the vector $\vec{a}-\vec{b}$ will be perpendicular to vector $\vec{b}$ when m is:

   A) 0 or 2;

   B) 0 or -2;

   C) $-1\pm\sqrt{3}$;

   D) $-1\pm\sqrt{5}$.

8. If $t=\tan\dfrac{\theta}{2}$ which of the following expressions is equivalent to $4\sin\theta+3\cos\theta+5$?

   A) $\dfrac{2(t+2)^2}{1-t^2}$;

   B) $\dfrac{(t+4)^2}{1-t^2}$;

   C) $\dfrac{2(t+2)^2}{1+t^2}$;

   D) $\dfrac{(t+4)^2}{1+t^2}$.

9. What is the value of $\cos^{-1}(\sin\alpha)$, where $\dfrac{\pi}{2}<\alpha<\pi$?

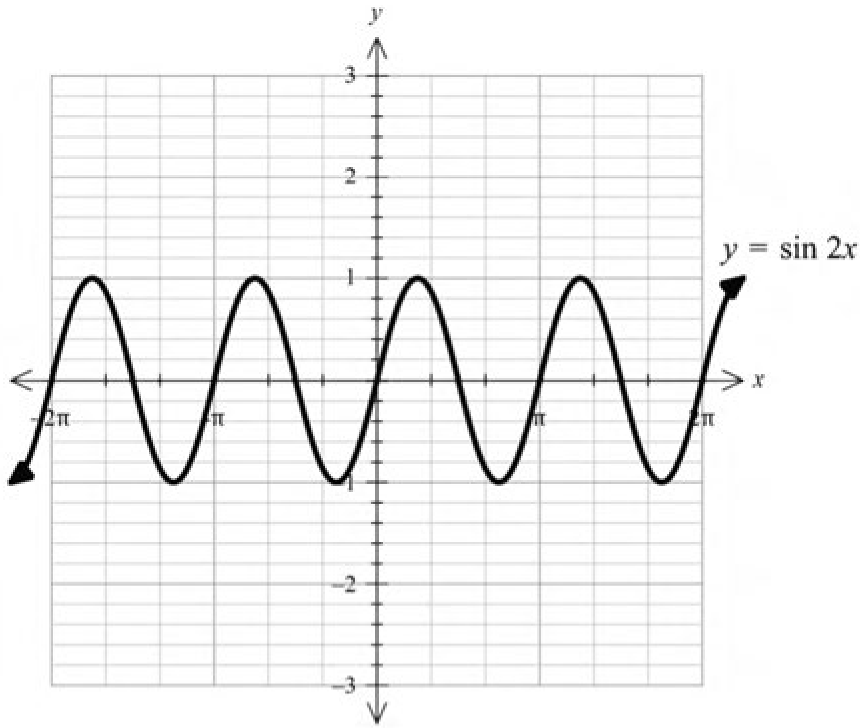
   A) $\pi-\alpha$;

   B) $\dfrac{\pi}{2}-\alpha$;

   C) $\alpha-\dfrac{\pi}{2}$;

   D) $\alpha+\dfrac{\pi}{2}$.

10. The function $y = \sin 2x$ is shown in the diagram.



If this function is transformed using steps I, II and III as below:

I.	Reflected about the x-axis
II.	Vertically translated 1 unit down
III.	Dilated horizontally by a scale factor of 2.

Which equation would represent the transformed function?

A) $y = -(\sin x + 1)$;

B) $x = 2(\sin(2y) - 1)$;

C) $y = -(\sin(4x)) + 2$;

D) $y = -2\sin(2x) - 1$.

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks) Use the Question 11 Writing Booklet

(a) Solve for x:

$$\frac{1-2x}{1+x} \geq 1.$$

3

(b)

(i) Express $\sin 2x + \sqrt{3}\cos 2x$ in the form of $R\sin(2x+\alpha)$. **2**

(ii) Find the greatest value of $\sin 2x + \sqrt{3}\cos 2x$ and the values of x, $0 \leq x \leq 2\pi$, at which it occurs. **2**

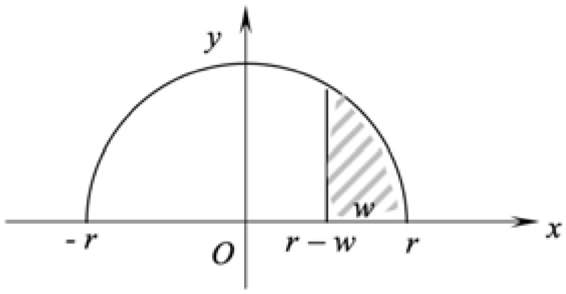
(iii) Hence, or otherwise, solve $\sin 2x + \sqrt{3}\cos 2x = 1$, $0 \leq x \leq 2\pi$ **3**

(c) By using substitution $u = \tan^{-1} x$ evaluate $\displaystyle\int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx$ **2**

(d) $P(x)$ is an odd polynomial of degree 3. It has $(x+4)$ as a factor and, when it is divided by $(x-3)$, the remainder is 21. Find $P(x)$. **3**

Question 12. (15 marks) Use the Question 12 Writing Booklet

(a) The semi-circle $y = \sqrt{r^2 - x^2}$ is show on diagram. **2**



The shaded area of thickness w is rotated about the x-axis to form the volume of a 'cap'. Show that the volume of the solid of revolution V is given by

$$V = \frac{\pi}{3}(3r - w)w^2.$$

(b) Consider the equation $\cos x - 2\sin x = 1$, for $-\pi \le x \le \pi$.

(i) Show that the equation can be written as $t^2 + 2t = 0$, where $t = \tan\frac{x}{2}$. **2**

(ii) Hence, solve the equation for $-\pi \le x \le \pi$, giving solutions to the 2 decimal places where necessary. **2**

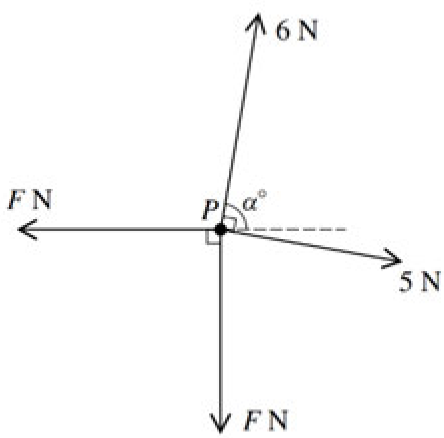
(c) (i) By equating coefficients, find the values of P and Q in the identity

$$P(2\sin x + \cos x) + Q(2\cos x - \sin x) \equiv 7\sin x + 11\cos x$$ **2**

(ii) Hence, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx$$ **2**

(d)



A particle P is in equilibrium under the action of four forces of magnitude 6 N, 5 N, F N and F N acting in the directions shown in the diagram on the right.

(i) By considering the vertical components of the forces, find an expression for F in terms of α. **2**

(ii) Hence, determine the value of α, correct to the nearest degree. **3**

Question 13. (15 marks) Use the Question 13 Writing Booklet

(a) (i) Prove that

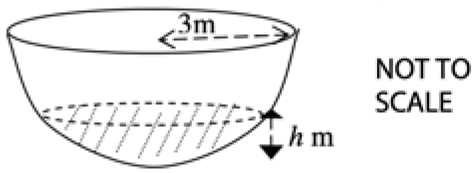
$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$ **2**

(iii) Hence, or otherwise, evaluate

$$\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$$ **2**

(b)

A hemi-spherical bowl has a radius of 3 m. Oil is poured in at a constant rate of $\frac{\pi}{3}$ m^3/min.



(i) Show that, when the depth of the oil is h metres, the volume of oil is:

$$V = \frac{\pi}{3}(9h^2 - h^3)$$ **2**

(ii) Show that the depth of the oil in the bowl after 8 minutes is 1 meter? **3**

(iii) At what rate is h increasing at this time? **2**

(c) Prove, by the method of mathematical induction that:

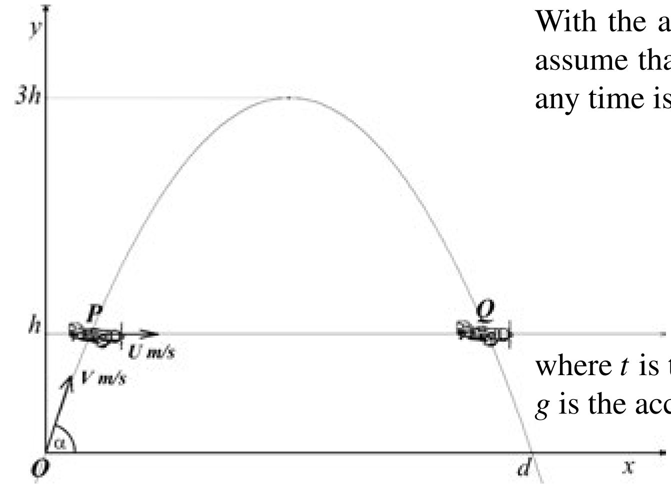
$$\sin q + \sin 3q + \sin 5q + ... + \sin(2n-1)q = \frac{1 - \cos 2nq}{2\sin q}, \text{ for } n = 1, 2, 3, ...$$ **4**

Question 14. (15 marks) Use the Question 14 Writing Booklet

(a) If $f(x) = 2 - \sqrt{x}$, $x \geq 0$, and $g(x) = (x-2)^2$ for all x, find the values of x for which **3**

$$f\big(g(x)\big) = g\big(f(x)\big)$$

(b) An enemy plane is flying horizontally at height h metres with speed U m/s.
When it is at point P, a ground rocket is fired towards it from the origin O with speed V m/s and angle of elevation α.
The rocket misses the plane, passing too late through the point P. However, it goes on to reach a maximum height of $3h$ metres and then on its descent strikes the plane at Q.



With the axes as shown in the diagram, you may assume that the position vector, $\vec{s}$, of the rocket at any time is given by:

$$\overrightarrow{s(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath},$$

$$x(t) = vt\cos\alpha$$

$$y(t) = -\frac{1}{2}gt^2 + vt\sin\alpha$$

where t is the time in seconds after firing and g is the acceleration due to gravity.

(i) Show that the initial vertical velocity component ($v\sin\alpha$) of the rocket's speed equals **2**

$$\sqrt{6gh}.$$

(ii) If the rocket had not struck the plane at Q, it would have returned to the x-axis at a distance d metres from O. Show that the horizontal component ($v\cos\alpha$) of the rocket's speed equals **2**

$$\frac{gd}{2\sqrt{6gh}}$$

(iii) Show that Cartesian equation of the trajectory of the rocket is **2**

$$y = \frac{12hx}{d}\left(1 - \frac{x}{d}\right).$$

(iv) If the horizontal component of the rocket's speed is $100(3+\sqrt{6})$ m/s, find the time taken by the rocket to strike the plane at Q, in terms of d. **2**

(v) Find the speed of the enemy plane. **1**

(c) Given the binomial expansion of:

$$(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$
$$(1+x)^{n+1} = b_0 + b_1x + b_2x^2 + \ldots + b_{n+1}x^{n+1}$$

and knowing $(1+x)^{n+1} = (1+x)^n(1+x)$, find the expression, **in terms of n only**, of: **3**

$$\frac{1}{a_0a_1a_2\ldots a_n} \times (a_o + a_1)(a_1+a_2)\ldots(a_{n-1}+a_n) \text{ for } n = 1, 2, 3, \ldots$$

End of Examination

MC

Q1 $\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\frac{x}{6} + C$ or $\arcsin\frac{x}{6}+C$ $\Rightarrow$ C

Q2 $\left(3x^2 - \frac{1}{2x}\right)^8$, $T_{k+1} = \binom{n}{k} \text{I}^{\,n-k}\ \text{II}^{\,k}$

$\text{I} \downarrow$ 1. term

II 2. term

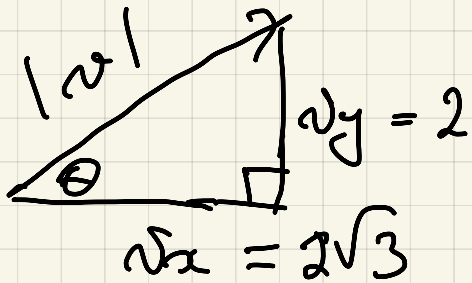
$$T_5 = \binom{8}{4}\left(3x^2\right)^{8-4}\left(-\frac{1}{2x}\right)^4$$

$$= \binom{8}{4}\frac{3^4}{2^4}(-1)^4\ x^8\ x^{-4}$$

$$= \binom{8}{4}\left(\frac{3}{2}\right)^4(-1)^4\ x^4$$

$$= \binom{8}{4}\left(\frac{3}{2}\right)^4 x^4 \Rightarrow \text{A}$$

Q3 $\underset{\sim}{v} = \begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix}$



$$|\underset{\sim}{v}|^2 = (2\sqrt{3})^2 + 2^2$$

$$= 4\times 3 + 4$$

$$= 16$$

$\therefore |\underset{\sim}{v}| = 4$

$$\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$\theta = 30^\circ \Rightarrow$ A

Q4. $f(x) = e^{x+2}$ $f^{-1}(x) = ?$

$x = e^{y+2}$

$\ln x = y+2$

$y = \ln x - 2 \Rightarrow$ C

Q6. $P(x) = 4x^3 + 13x^2 + ax$ A

$(x+a) = (x-(-a))$

$P(-a) = 0$

$= -4a^3 + 13a^2 - a^2 = 0$

$\Rightarrow -4a^3 + 12a^2 = 0$

$-4a^2(a-3) = 0$

$\therefore a = 3 \Rightarrow$ A

Q5. $\frac{dr}{dt} = +5 \frac{cm}{min}$

$\frac{dV}{dt}\left[\frac{cm^3}{min}\right] = ?$, $r = 6cm$

$V_{sphere} = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$

$= \frac{4}{\cancel{3}}\pi \times \cancel{3} \times 6^2 \times 5 \frac{cm^3}{min} = 720\pi \frac{cm^3}{min} \Rightarrow$ D

Q7. $\vec{a} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} m \\ -1 \end{pmatrix}$

$\vec{a} - \vec{b} = (-2-m)\hat{i} + (-1+1)\hat{j} = \begin{pmatrix} -2-m \\ 0 \end{pmatrix}$

$(\vec{a} - \vec{b}) \perp \vec{b}$, if $(\vec{a} - \vec{b}) \cdot \vec{b} = 0$

$(-2-m) \cdot m + (-1) \times 0 = 0$

$-2m - m^2 = 0$

$m(2+m) = 0 \Rightarrow m = 0$ or $m = -2$

$\Downarrow$

B

Q8. $t = \tan\frac{\theta}{2}$

$4\sin\theta + 3\cos\theta + 5 = ?$

$4\,\frac{2t}{1+t^2} + 3\,\frac{1-t^2}{1+t^2} + 5 =$

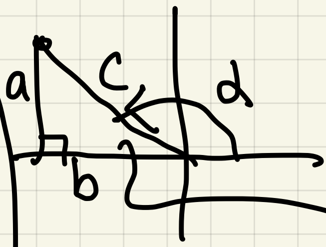
$= \frac{8t+3-3t^2+5+5t^2}{1+t^2}$

$= \frac{2t^2+8t+8}{1+t^2}$

$= \frac{2(t^2+4t+4)}{1+t^2}$

$= \frac{2(t+2)^2}{1+t^2} \Rightarrow$ (C)

Q9. $\cos^{-1}(\sin\alpha))$ $\quad \frac{\pi}{2}<\alpha<\pi \Rightarrow$ 2nd quad.



$\sin\alpha>0$

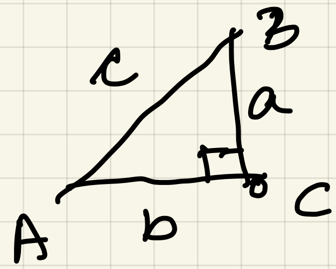
$\sin\alpha = \sin(180-\alpha)$

$= \frac{a}{c}$

$\therefore \cos^{-1}(\sin\alpha) =$

$= \cos^{-1}(\frac{a}{c})$

$= \angle ABC$



$\angle BAC = \pi-\alpha$

$\therefore \angle ABC = \frac{\pi}{2}-(\pi-\alpha)$

$= -\frac{\pi}{2}+\alpha$

$= \alpha+\frac{\pi}{2} \Rightarrow$ (C)

Or use test cases:

$\cos^{-1}(\sin\frac{5\pi}{6}) = \cos^{-1}(\frac{1}{2})$

$= \frac{\pi}{3}$

if $\alpha = \frac{5\pi}{6}$, $\quad \pi-\alpha = \frac{\pi}{6}$ not A)

$\frac{\pi}{2}-\alpha = \frac{-2\pi}{6} = \frac{-\pi}{3}$ not B)

$\alpha+\frac{\pi}{2} = \frac{8\pi}{6}$ not (D)

$\therefore$ (C)

Q_{10} $y = \sin 2x$

$I: y = -\sin 2x$

$II: y = -\sin 2x - 1$

$III: y = -\sin(2x \times \frac{1}{2}) - 1$

$= -\sin x - 1$

$= -(\sin x + 1) \Rightarrow$ A

Q_1 C

Q_2 A

Q_3 A

Q_4 C

Q_5 D

Q_6 A

Q_7 B

Q_8 C

Q_9 C

Q_{10} A

Q11 a) $\dfrac{1-2x}{1+x} \geq 1 \quad , \quad x \neq -1$

$\dfrac{1-2x}{1+x} - 1 \geq 0$

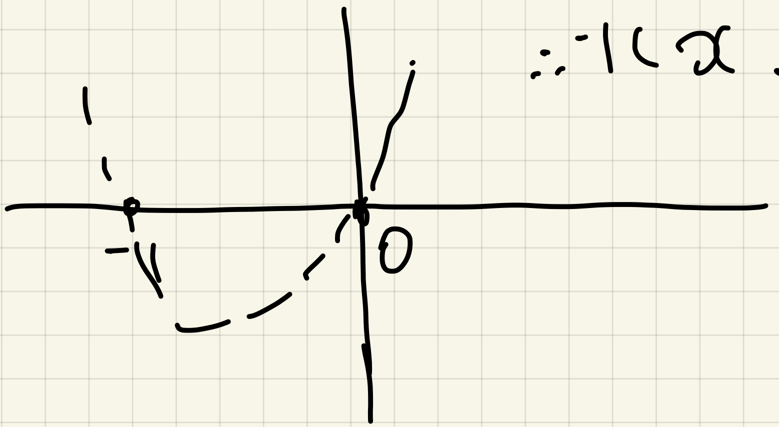
$\dfrac{1-2x-1-x}{1+x} \geq 0$ 1

$\dfrac{-3x}{1+x} \geq 0$

$\dfrac{3x}{1+x} \leq 0 \quad \Big/ \; x(1+x)^2$

$3x(1+x) \leq 0$ 1 + 1

$\therefore -1 < x \leq 0$



Q11 b)

i) $\sin 2x + \sqrt{3}\cos 2x$

$R\sin(2x+\alpha)$

$R\sin 2x\cos\alpha + R\cos 2x\sin\alpha =$

$= \sin 2x + \sqrt{3}\cos 2x$

$\therefore \left.\begin{array}{l} R\cos\alpha = 1 \\ R\sin\alpha = \sqrt{3} \end{array}\right\} \quad R^2 = 4 \Rightarrow R = 2$ 1

$\tan\alpha = \sqrt{3}$

$\therefore \alpha = \frac{\pi}{3}$ 1

$\therefore \sin 2x + \sqrt{3}\cos 2x = 2\sin\left(2x + \frac{\pi}{3}\right)$

ii) greatest value for $2\sin\left(2x+\frac{\pi}{3}\right) = 2 \times 1$

1 $\therefore \underline{\underline{2}}$ for $0 \leq x \leq 2\pi$

$0 \le 2x \le 4\pi$

$\frac{\pi}{3} \le 2x + \frac{\pi}{3} \le 4\pi + \frac{\pi}{3}$

$\frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3}$

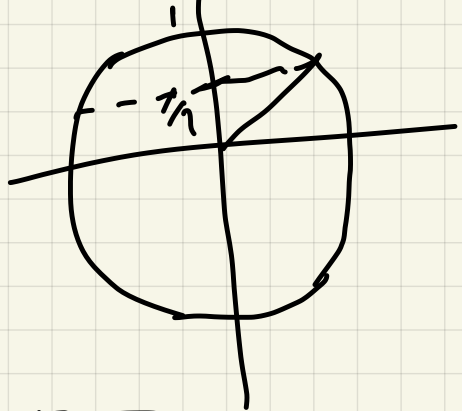
$\therefore\ 2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{5\pi}{2}$

$2x = \frac{\pi}{2} - \frac{\pi}{3}, \frac{5\pi}{2} - \frac{\pi}{3}$

$2x = \frac{3\pi - 2\pi}{6}, \frac{15\pi - 2\pi}{6}$

$x = \frac{\pi}{12}, \frac{13\pi}{12}$ (1)

$2x + \frac{\pi}{3} = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$ (1)



$2x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, 4\pi + \frac{\pi}{6}$

$2x = \frac{\pi}{6} - \frac{\pi}{3}, \frac{5\pi}{6} - \frac{\pi}{3}, \frac{13\pi}{6} - \frac{\pi}{3}, \frac{17\pi}{6} - \frac{\pi}{3}, \frac{25\pi}{6} - \frac{\pi}{3}$

$2x = \frac{-\pi}{6}, \frac{3\pi}{6}, \frac{11\pi}{6}, \frac{15\pi}{6}, \frac{23\pi}{6}$

$x = \frac{-\pi}{12}, \frac{3\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$ (1)

iii) $\sin 2x + \sqrt{3}\cos 2x = 1, \quad 0 \le x \le 2\pi$

$2\sin\left(2x + \frac{\pi}{3}\right) = 1$

$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore\ x = \frac{3\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$ (1)

Q11 c) $u = \tan^{-1}x$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$$

$$du = \frac{dx}{1+x^2}$$ ①

$$\int e^u\,du = e^u + C$$

$$= e^{\tan^{-1}x} + C$$ ①

Q11 d) odd $P(-x) = -P(x)$

$$P(x) = ax^3 + bx^2 + cx + d$$

$$P(-x) = -ax^3 + bx^2 - cx + d = -ax^3 - bx^2 - cx - d$$

$b = -b \Rightarrow b = 0$

$d = -d \Rightarrow d = 0$

$\therefore P(x) = ax^3 + cx$ ①

$$P(x) = ax^3 + cx = x(ax^2 + c)$$

$$P(-4) = -4(16a + c) = 0 \Rightarrow c = -16a$$

$$P(3) = 3(9a + c) = 21 \Rightarrow 9a + c = 7$$

$\therefore$ $9a - 16a = 7$

$-7a = 7$

$a = -1$ ①

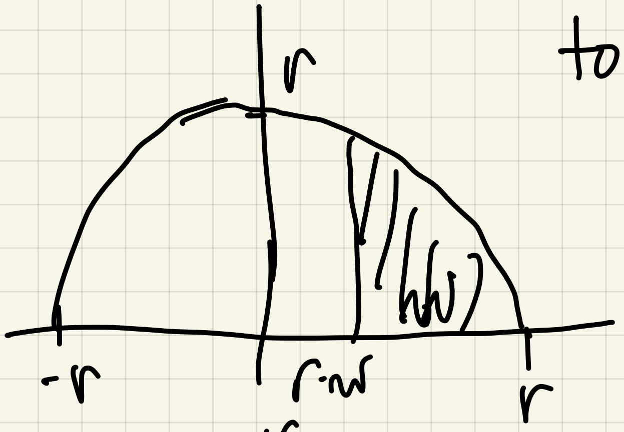
$\therefore c = (-16) \times (-1) = 16$ ①

$\therefore P(x) = -1x^3 + 16x$

$= -x^3 + 16x$

$= 16x - x^3$

Q12 a) $y = \sqrt{r^2 - x^2}$



to show

$$V = \frac{\pi}{3}(3r-w)w^2$$

$$V = \pi \int_{r-w}^{r} y^2 \, dx$$

$$= \pi \int_{r-w}^{r} (r^2 - x^2) \, dx$$ 1

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-w}^{r} =$$

$$= \pi \left[r^2 (r - (r-w)) - \frac{1}{3} (r^3 - (r-w)^3) \right]$$

$$= \pi \left[\underline{r^2 w} - \frac{1}{3} (\cancel{r^3} - \cancel{r^3} + \underline{3r^2 w} - 3rw^2 + w^3) \right]$$

0

$$= \pi \left[rw^2 - \frac{1}{3} w^3 \right]$$

$$= \frac{\pi}{3} w^2 \left[3r - w \right]$$

$$\therefore V = \frac{\pi}{3}(3r-w)w^2$$ 1

Q12 b) $\cos x - 2\sin x = 1 \quad -\pi \leq x \leq \pi \quad \Rightarrow -\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{\pi}{2}$

from ref. sheet: $\sin x = \frac{2t}{1+t^2}$

i) $t = \tan\frac{x}{2}$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$$\frac{1-t^2}{1+t^2} - 2\frac{2t}{1+t^2} = 1 \quad (1)$$

$$\frac{1-t^2-4t}{1+t^2} = 1$$

$$-t^2-4t \cancel{+1} = \cancel{1}+t^2$$

$$-2t^2-4t=0$$

$$2t^2+4t=0$$

$$t^2+2t=0 \quad (1)$$

ii) $t(t+2)=0$

$t=0$ or $t=-2$

$\therefore \tan\frac{x}{2} = 0$ or -2 (1)

$$\frac{x}{2} = \arctan 0 \text{ or } \arctan(-2)$$

$$x = 0 \text{ or } 2\arctan(-2)$$

$$x = 0 \text{ or } -2.2143 \quad (1)$$

Q12c) i)

$$P(2\sin x + \cos x) + Q(2\cos x - \sin x) = 7\sin x + 11\cos x$$

$$2P\sin x + P\cos x + 2Q\cos x - Q\sin x = 7\sin x + 11\cos x$$

$$\therefore \quad 2P - Q = 7 \quad /\times 2$$

$$P + 2Q = 11$$

$$5P = 25$$

$$P = 5 \Rightarrow Q = 2P - 7 = 10 - 7 = 3$$

$\therefore$ 1 1

$P = 5$, $Q = 3$

ii)

$$\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{5(2\sin x + \cos x)}{2\sin x + \cos x}\,dx +$$

$$3\int_0^{\frac{\pi}{2}} \frac{2\cos x - \sin x}{2\sin x + \cos x}\,dx = 5\int_0^{\frac{\pi}{2}} dx + 3\int_0^{\frac{\pi}{2}} \frac{f'(x)}{f(x)}\,dx =$$

1

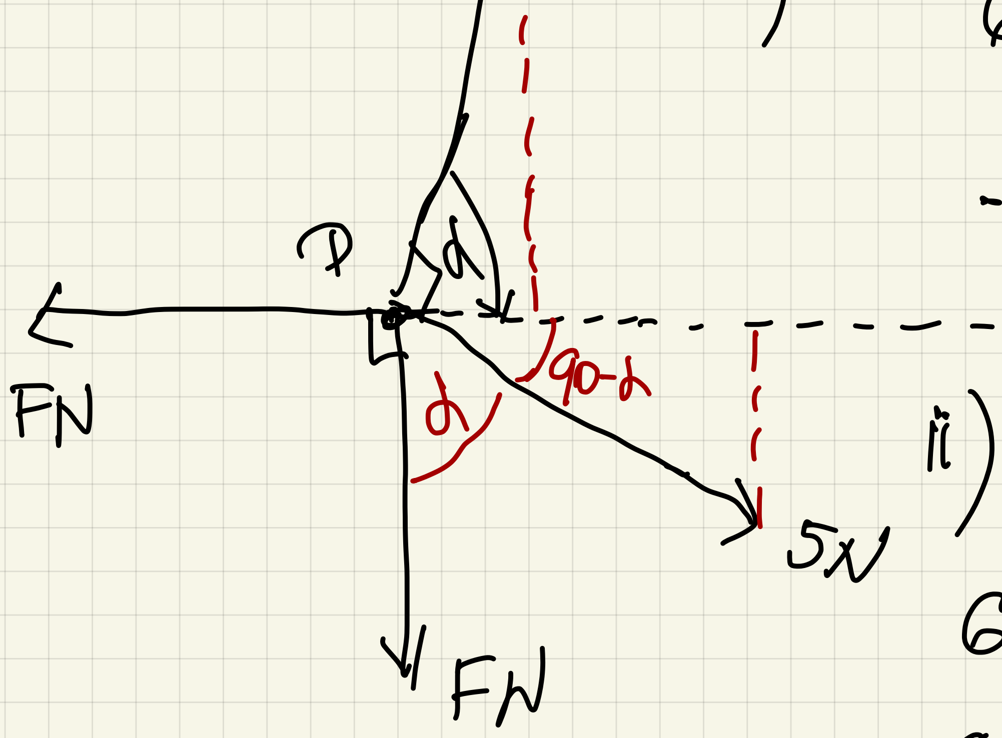
Q12c) ii) continued

$$5\left(\frac{\pi}{2}-0\right)+3\ln\left|2\sin x+\cos x\right|\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{5\pi}{2}+3\Big[\ln|2+0|-\ln 1\Big]$$

$$=\frac{5\pi}{2}+3\ln 2 \quad 1$$

Q12d)



i) for vertical components 1

$$6\sin\alpha - 5\sin(90-\alpha) - F + 0 = 0$$

$$\therefore F = 6\sin\alpha - 5\cos\alpha \quad 1$$

ii) for horizontal components

$$6\cos\alpha + 5\cos(90-\alpha) + 0 - F = 0 \quad 1$$

$$\left.\begin{aligned} 6\cos\alpha + 5\sin\alpha &= F \\ \text{from (i)}: \quad 6\sin\alpha - 5\cos\alpha &= F \end{aligned}\right\} -$$

$$11\cos\alpha - \sin\alpha = 0 \quad 1$$

$$\tan\alpha = 11$$

$$\alpha = 84.80557^\circ \approx 85^\circ \quad 1$$

Q13 a)

i) prove $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

let $y = \cos^{-1}(-x)$

$\cos y = -x$

$\cos(\pi - y) = -\cos y$ 1

$= -(-x)$

$= x$

$\pi - y = \cos^{-1}(x)$

$\therefore \pi - \cos^{-1}(-x) = \cos^{-1}(x)$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ 1

ii) $\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{\pi}{2}-\left(-\frac{\pi}{3}\right)\right)\right)$ 1 using $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$ Restriction of domain for $\cos^{-1}(\cos(x)) = x, \quad x\in[0,\pi]$

$= \frac{5\pi}{6}$ 1 , as $\frac{5\pi}{6}\in[0,\pi]$

Or

$\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$

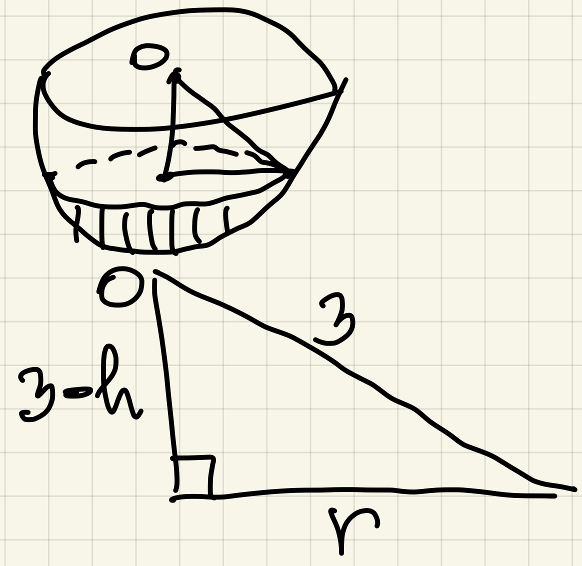
$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ using (i) part 1

$= \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$ 1

Q13 b) $R = 3m$, $\frac{dV}{dt} = \frac{\pi}{3}\ \frac{m^3}{min}$



i) to show $V = \frac{\pi}{3}(9h^2 - h^3)$

using Pythagoras's

$$3^2 = (3-h)^2 + r^2$$

$$r^2 = 9 - (3-h)^2$$

$$= 9 - 9 + 6h - h^2$$

$$= 6h - h^2 \quad (1)$$

Volume: $V = \pi \int_0^h (6h - h^2)\, dh$

or

$$= \pi \left[6\frac{h^2}{2} - \frac{h^3}{3} \right]_0^h$$

$$= \pi \left[3h^2 - \frac{h^3}{3} \right] = \frac{\pi}{3}\left[9h^2 - h^3 \right] m^3 \quad (1)$$

Alternmatively:

Rotated $x^2+y^2=9$ around y-axis

between 3 and 3-h

$\therefore x^2 = 9-y^2$

$$V = \pi\int_{3-h}^{3} x^2\,dy$$

$$= \pi\int_{3-h}^{3} (9-y^2)\,dy$$

$$= \pi\left[9y - \frac{y^3}{3}\right]_{3-h}^{3}$$

$$= \pi\left[27 - \frac{27}{3} - \left[(27-9h) - \frac{(3-h)^3}{3}\right]\right] =$$

$$\pi\left[\cancel{27} - 9 - \cancel{27} + 9h + \frac{1}{3}(27 - 27h + 9h^2 - h^3)\right]$$

$$= \pi\left[\cancel{-9} \cancel{+9h} \cancel{+9} \cancel{-9h} + 3h^2 - \frac{1}{3}h^3\right]$$

$$= \pi\left[3h^2 - \frac{1}{3}h^3\right]$$

$$= \frac{\pi}{3}\left[9h^2 - h^3\right]$$

ii) $h = ?$ $t = 8\,\text{min}$

$$\frac{dV}{dt} = \frac{\pi}{3} \frac{m^3}{\text{min}}$$

$$V = \frac{\pi}{3} t \frac{m^3}{\text{min}}, \text{ at } t = 8\,\text{min}$$

$$\Rightarrow V = \frac{8\pi}{3} \frac{m^3}{\text{min}} \quad 1$$

$$\frac{8\pi}{3} = \frac{\pi}{3}(9h^2 - h^3)$$

$$8 = 9h^2 - h^3$$

$$h^3 - 9h^2 + 8 = 0$$

$$(h-1)(h^2 - 8h - 8) = 0 \quad 1$$

$h - 1 = 0$ or $h^2 - 8h - 8 = 0$

$\therefore h = 1$ or $h = \dfrac{8 \pm \sqrt{64 - 4 \times (-8) \times 1}}{2} = \dfrac{8 \pm \sqrt{64 + 32}}{2} = \dfrac{8 \pm \sqrt{96}}{2}$

$$h^3 - 9h^2 + 8 = (h-1)(h^2 - 8h - 8)$$

$$\begin{array}{c|cccc}
1 & 1 & -9 & 0 & 8 \\
 & & 1 & -8 & -8 \\
\hline
 & 1 & -8 & -8 & \boxed{0}
\end{array}$$

$$= \frac{8 \pm \sqrt{8 \times 12}}{2}$$

$$= \frac{8 \pm \sqrt{8 \times 2 \times 6}}{2}$$

$$= \frac{8 \pm 4\sqrt{6}}{2}$$

$$= 4 \pm 2\sqrt{6}$$

as $0 \le h \le 3$

$\therefore\ h = 1$ only (1)

iii) $\frac{dh}{dt} = ?$

$\frac{dV}{dt} = \frac{\pi}{3}$ (given), $V = \frac{\pi}{3}[9h^2 - h^3]$ (proved)

$$\frac{dV}{dh} = \frac{\pi}{3}[18h - 3h^2]$$

$$= \pi[6h - h^2]$$

$$\frac{dV}{dt} = \frac{dV}{dh}\ \underline{\underline{\frac{dh}{dt}}}$$

$$\frac{\pi}{3} = \frac{dV}{dh} \times \frac{dh}{dt} \quad (1)$$

$$\frac{\pi}{3} = \pi[6h - h^2] \times \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{\pi}{3} \times \frac{1}{\pi[6h - h^2]}$$

$\frac{dh}{dt} = \frac{1}{3[6h - h^2]}$, when $h = 1\,m \Rightarrow \frac{dh}{dt} =$

$$\therefore \frac{dh}{dt} = \frac{1}{3[6h-h^2]}$$

for $h = 1$ m

$$\frac{dh}{dt} = \frac{1}{3[6\times1-1]} = \frac{1}{3\times5} = \frac{1}{15}\ \frac{m}{min}$$ (1)

$\therefore$ when oil is 1 m deep, h is increasing at a rate of $\frac{1}{15}$ meters per minute.

Q13 c) $\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta}$

Step 1 : $n=1$

$LHS = \sin\theta$ $\qquad RHS = \frac{1-\cos 2\theta}{2\sin\theta}$

$$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta}$$

1

$$= \frac{2\sin^2\theta}{2\sin\theta}$$

$$= \sin\theta = LHS$$

$\therefore$ true for $n=1$

Step 2 :

Assume true $n=k$: $\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$

To prove true for $n=k+1$:

$$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2k+1)\theta = \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

$$LHS = \sin g + \sin 3g + \sin 5g + \cdots + \sin(2k-1)g + \sin(2k+1)g$$

$$= \frac{1-\cos 2kg}{2\sin g} + \sin(2k+1)g \quad \text{(by assumption)}$$

$$= \frac{1-\cos 2kg + 2\sin g \sin(2k+1)g}{2\sin g}$$

$$= \frac{1-\cos[(2k+1)g - g] + 2\sin g \sin(2k+1)g}{2\sin g} \qquad 1$$

$$= \frac{1-[\cos(2k+1)g\cos g + \sin(2k+1)g \sin g] + 2\sin g \sin(2k+1)g}{2\sin g}$$

$$= \frac{1-[\cos(2k+1)g\cos g - \sin(2k+1)g \sin g]}{2\sin g}$$

$$= \frac{1-\cos[(2k+1)g + g]}{2\sin g} = \frac{1-\cos[2kg + g + g]}{2\sin g} = \frac{1-\cos(2k+2)g}{2\sin g} \qquad 1$$

$$= \frac{1-\cos 2(k+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

Step 3 :- If statement true for $n=k$ it is true for $n=k+1$.

Since true for $n=1, \ldots, n=k$, and $n=k+1$, it is true for any n.

1

Q14a)

$f(x) = 2 - \sqrt{x} \quad , x \geq 0$

$g(x) = (x-2)^2 , \text{ for all } x$

i) $f(g(x)) = g(f(x))$

$D_{f(x)} : [0, +\infty)$

$R_{f(x)} : (-\infty, 2]$

$f(g(x)) = f\left((x-2)^2\right) = 2 - \sqrt{(x-2)^2}$

$\sqrt{(x-2)^2} = \begin{cases} x-2, & x \geq 2 \\ -x+2, & x < 2 \end{cases}$

1

$\therefore f(g(x)) = \begin{cases} 2-(x-2) = 4-x, & x \geq 2 \\ 2-(-x+2) = x, & x < 2 \end{cases} = \begin{cases} 4-x, & x > 2 \\ 4-2 = 2 = x, & x = 2 \\ x, & x < 2 \end{cases}$

$$g(f(x)) = g(2-\sqrt{x}) = ((2-\sqrt{x})-2)^2 = (-\sqrt{x})^2 = x$$

① for all $x \geq 0$

$\therefore\ [0,+\infty) \cap (-\infty, 2] = [0, 2]$

$\therefore\ f(g(x)) = x = g(f(x))$ for $0 \leq x \leq 2$ ①

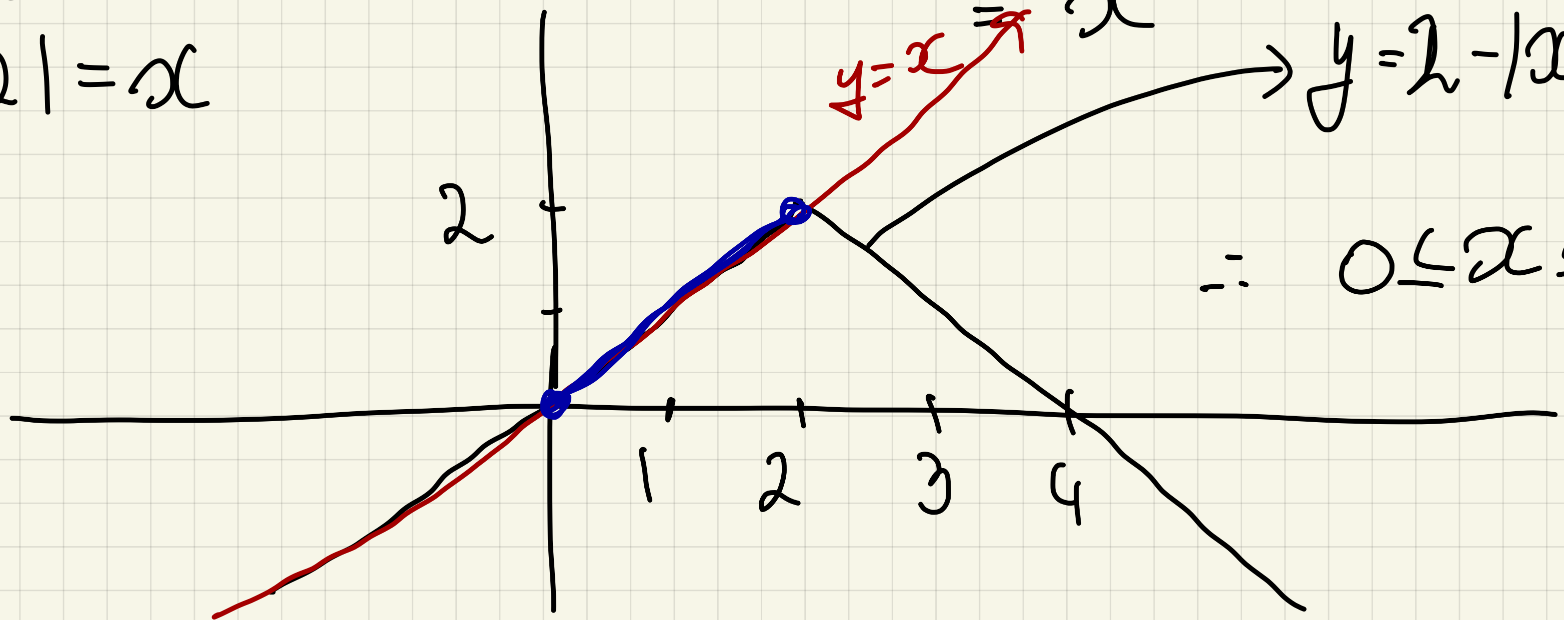
AH :

$$f(g(x)) = f((x-2)^2) = 2 - \sqrt{(x-2)^2} = 2 - |x-2|$$

$$g(f(x)) = g(2-\sqrt{x}) = (2-\sqrt{x}-2)^2 = |x|, \text{ but } x \geq 0$$

$= x$

$2 - |x-2| = x$



$\therefore\ 0 \leq x \leq 2$

Q14 b) $x = vt\cos\alpha$ (1)

$y = -\frac{1}{2}gt^2 + vt\sin\alpha$ (2)

i) $v\sin\alpha = v_y = \sqrt{6gh}$ to show

$\dot{y} = -\frac{1}{2} \times 2gt + v\sin\alpha = -gt + v\sin\alpha$

$\dot{y} = 0$ for maximum height

$0 = -gt + v\sin\alpha$

$gt = v\sin\alpha$

$t = \frac{v\sin\alpha}{g}$ into (2) for $y_{max} - 3h$ → ①

$$y_{max} = -\frac{1}{2}g\left(\frac{v\sin\alpha}{g}\right)^2 + v\frac{v\sin\alpha}{g}\sin\alpha = -\frac{1}{2}\frac{v^2\sin^2\alpha}{g} + \frac{v^2\sin^2\alpha}{g} =$$

$$y_{max} = \frac{1}{2}\frac{v^2 \sin^2 \alpha}{g}$$

$y_{max} = 3h$ (given)

$$3h = \frac{1}{2}\frac{v^2 \sin^2 \alpha}{g}$$

$$6gh = v^2 \sin^2 \alpha$$

$$\therefore v \sin \alpha = \sqrt{6gh}$$

ii) $v \cos \alpha = \dfrac{gd}{2\sqrt{6gh}}$ to show

Range $= d$, when $y = 0$

$$0 = t\left(v \sin \alpha - \tfrac{1}{2} g t\right)$$

$t = 0$ or $v \sin \alpha - \frac{1}{2} g t = 0$

at the start

at the end of flight

$$\therefore \tfrac{1}{2} g t = v \sin \alpha$$

$$t = \frac{2v \sin \alpha}{g}$$ time of flight

so

Range $= vt\cos \alpha$

$$= v \times \frac{2v \sin \alpha}{g} \times \cos \alpha$$

$$\text{Range} = \frac{v^2}{g} 2 \sin \alpha \cos \alpha$$

Range $= d$

$$\frac{v^2}{g} 2\sin\alpha\cos\alpha = d$$

$$\underline{v\cos\alpha} \times 2\frac{v}{g}\sin\alpha = d$$

$$v\cos\alpha = d \times \frac{1}{2}\frac{g}{v\sin\alpha}$$

now $v\sin\alpha = \sqrt{6gh}$

$$\therefore v\cos\alpha = \frac{gd}{2} \times \frac{1}{\sqrt{6gh}} \quad (1)$$

$$= \frac{gd}{2\sqrt{6gh}}$$

iii) to show $y = \frac{12hx}{d}\left(1 - \frac{x}{d}\right)$

$$t = \frac{x}{v\cos\alpha} \quad \text{sub}$$

$$y = -\frac{1}{2}g\left(\frac{x^2}{v^2\cos^2\alpha}\right) + v\frac{x}{v\cos\alpha}\sin\alpha \quad (1)$$

$$= -\frac{1}{2}g\,\frac{x^2}{\frac{g^8 d^2}{24gh}} + 12\frac{h}{d}x$$

$$\frac{v\sin\alpha}{v\cos\alpha} = \frac{\sqrt{6gh}}{\frac{gd}{2\sqrt{6gh}}}$$

$$= \frac{12gh}{gd}$$

$$= 12\frac{h}{d} \quad (*)$$

$$\therefore \quad y = -12\frac{hx^2}{d^2} + 12\frac{h}{d}x$$

$$= \frac{12hx}{d}\left(1 - \frac{x}{d}\right) \quad (1)$$

iv) $v_x = 100(3+\sqrt{6})\frac{m}{s} \Rightarrow x = v_x t$

$t = f(d) = ?$

---

at Q $\quad y = h$

from (iii) $\quad y = \frac{12hx}{d}\left(1 - \frac{x}{d}\right)$

$$\cancel{h} = \frac{12\cancel{h}x}{d}\left(1 - \frac{x}{d}\right)$$

$$1 = 12\frac{x}{d}\left(1-\frac{x}{d}\right)$$

$$d^2 = 12xd\left(1-\frac{x}{d}\right)$$

$$d^2 = 12xd - 12x^2$$ 1

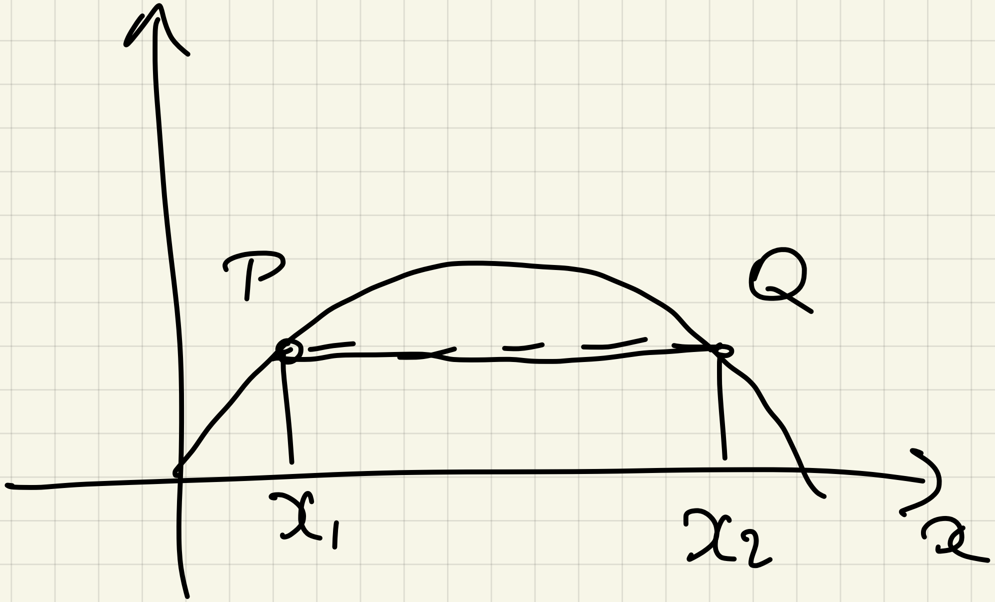
$$12x^2 - 12xd + d^2 = 0$$

$$x_{1,2} = \frac{12d \pm \sqrt{144d^2 - 4\times 12\times d^2}}{24}$$

$$= \frac{12d \pm \sqrt{96d^2}}{24}$$

$$= \frac{d}{2} \pm \frac{4d\sqrt{6}}{24}$$

$$= \frac{d}{2} \pm \frac{d\sqrt{6}}{6} = \frac{d(3\pm\sqrt{6})}{6}$$



at Q: $x = \frac{3+\sqrt{6}}{6}d$

$$x = v_x t$$

$$\frac{\cancel{3+\sqrt{6}}}{6}d = 100(\cancel{3+\sqrt{6}})t$$

$$\therefore\ t = \frac{d}{600}$$ 1

v) $u = ?$ just horizontal component $\Rightarrow x = ut$

$$PQ = \text{distance} = x_2 - x_1 = \frac{3+\sqrt{6}}{6}d - \frac{3-\sqrt{6}}{6}d$$

$$\therefore PQ = \frac{2\sqrt{6}}{6}d = \frac{\sqrt{6}}{3}d$$

$$\therefore \frac{\sqrt{6}}{3}d = u \times \frac{d}{600}$$

$$\therefore u = \frac{\frac{\sqrt{6}}{3}\cancel{d}}{\frac{\cancel{d}}{600}} = \frac{600\sqrt{6}}{3} = 200\sqrt{6}$$

$$u \approx 490 \ \frac{m}{s}$$

1

Q 14 c)

$b_k = a_{k-1} + a_k$ from Pascal Triangle

$$\frac{1}{a_0 a_1 a_2 \cdots a_n} \times (a_0+a_1)(a_1+a_2)(a_2+a_3)\cdots(a_{n-1}+a_n)$$

$n = 1, 2, 3, \ldots$

$$= \frac{1}{a_0} \times \frac{1}{a_1 a_2 \cdots a_n} \times b_1\ b_2\ b_3 \cdots b_n \quad 1 \quad , \quad a_0 = 1$$

$$= 1 \times \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}$$

now $$\frac{b_k}{a_k} = \frac{{}^{n+1}C_k}{{}^{n}C_k} = \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{(n+1-k)!}_{(n-k+1)}\cancel{k!}} \times \frac{\cancel{(n-k)!}\ \cancel{k!}}{\cancel{n!}}$$

$$\frac{b_k}{a_k} = \frac{n+1}{n-k+1} \quad 1$$

$$\frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} = \frac{(n+1)(n+1)\cdots(n+1)}{n(n-1)(n-2)\cdots 1}$$

$$= \frac{(n+1)^n}{n!} \quad ①$$

$$\therefore \frac{1}{a_0 a_1 a_2 \cdots a_n} \times (a_0+a_1)(a_1+a_2)\cdots(a_{n-1}+a_n)$$

$$= \frac{(n+1)^n}{n!}$$